# Machine Learning

## A Probabilistic Perspective

Kevin P. Murphy

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This book is dedicated to Alessandro, Michael and Stefano,
and to the memory of Gerard Joseph Murphy.

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# *Preface*

## Introduction

With the ever increasing amounts of data in electronic form, the need for automated methods for data analysis continues to grow. The goal of machine learning is to develop methods that can automatically detect patterns in data, and then to use the uncovered patterns to predict future data or other outcomes of interest. Machine learning is thus closely related to the fields of statistics and data mining, but differs slightly in terms of its emphasis and terminology. This book provides a detailed introduction to the field, and includes worked examples drawn from application domains such as molecular biology, text processing, computer vision, and robotics.

## Target audience

This book is suitable for upper-level undergraduate students and beginning graduate students in computer science, statistics, electrical engineering, econometrics, or any one else who has the appropriate mathematical background. Specifically, the reader is assumed to already be familiar with basic multivariate calculus, probability, linear algebra, and computer programming. Prior exposure to statistics is helpful but not necessary.

## A probabilistic approach

This books adopts the view that the best way to make machines that can learn from data is to use the tools of probability theory, which has been the mainstay of statistics and engineering for centuries. Probability theory can be applied to any problem involving uncertainty. In machine learning, uncertainty comes in many forms: what is the best prediction (or decision) given some data? what is the best model given some data? what measurement should I perform next? etc.

The systematic application of probabilistic reasoning to all inferential problems, including inferring parameters of statistical models, is sometimes called a Bayesian approach. However, this term tends to elicit very strong reactions (either positive or negative, depending on who you ask), so we prefer the more neutral term “probabilistic approach”. Besides, we will often use techniques such as maximum likelihood estimation, which are not Bayesian methods, but certainly fall within the probabilistic paradigm.

Rather than describing a cookbook of different heuristic methods, this book stresses a principled model-based approach to machine learning. For any given model, a variety of algorithms can often be applied. Conversely, any given algorithm can often be applied to a variety of models. This kind of modularity, where we distinguish model from algorithm, is good pedagogy and good engineering.

We will often use the language of graphical models to specify our models in a concise and intuitive way. In addition to aiding comprehension, the graph structure aids in developing efficient algorithms, as we will see. However, this book is not primarily about graphical models; it is about probabilistic modeling in general.

## A practical approach

Nearly all of the methods described in this book have been implemented in a MATLAB software package called **PMTK**, which stands for probabilistic modeling toolkit. This is freely available from `pmtk3.googlecode.com` (the digit 3 refers to the third edition of the toolkit, which is the one used in this version of the book). There are also a variety of supporting files, written by other people, available at `pmtksupport.googlecode.com`. These will be downloaded automatically, if you follow the setup instructions described on the PMTK website.

MATLAB is a high-level, interactive scripting language ideally suited to numerical computation and data visualization, and can be purchased from `www.mathworks.com`. Some of the code requires the Statistics toolbox, which needs to be purchased separately. There is also a free version of Matlab called **Octave**, available at `http://www.gnu.org/software/octave/`, which supports most of the functionality of MATLAB. Some (but not all) of the code in this book also works in Octave. See the PMTK website for details.

PMTK was used to generate many of the figures in this book; the source code for these figures is included on the PMTK website, allowing the reader to easily see the effects of changing the data or algorithm or parameter settings. The book refers to files by name, e.g., `naiveBayesFit`. In order to find the corresponding file, you can use two methods: within Matlab you can type `which naiveBayesFit` and it will return the full path to the file; or, if you do not have Matlab but want to read the source code anyway, you can use your favorite search engine, which should return the corresponding file from the `pmtk3.googlecode.com` website.

Details on *how to use* PMTK can be found on the website, which will be udpated over time. Details on the *underlying theory* behind these methods can be found in this book.

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